§1. Clifford algebra

V : n-dimensional real inner space. $\{e_1, \cdots, e_n\}$ orthonormal basis of V.

$\langle \cdot, \cdot \rangle$: inner product of V. norm: $\|v\|^2 := \langle v, v \rangle$.

Def. Clifford algebra. Set $\otimes V = \bigoplus_{k \geq 0} \otimes^k V$. $Cl(V) = \otimes V / \mathcal{H}$. where $\mathcal{H}$ is the two-sided ideal generated by all elements of the form $v \otimes v + \|v\|^2$, $v \in V$.

Remark: $\mathcal{H} = \{\sum_{i=1}^{m} w_i \otimes (v_i \otimes v_i + \|v_i\|^2) \otimes z_i : w_i, z_i \in \otimes V, v_i \in V, 1 \leq i \leq m, m \in N_+\}$.

the quotient map: $\pi : \otimes V \to Cl(V)$ algebra homomorphism.

By definition, $\pi(v \otimes v + \|v\|^2) = \pi(v) \cdot \pi(v) + \|v\|^2 \cdot 1 = 0$, $\forall v \in V$.

$$\Leftrightarrow \quad \pi(v) \cdot \pi(w) + \pi(w) \cdot \pi(v) + 2\langle v, w \rangle = 0. \quad \forall v, w \in V.$$

$$\Leftrightarrow \quad \pi(e_i) \cdot \pi(e_j) + \pi(e_j) \cdot \pi(e_i) = 0 \quad i \neq j$$

$$\pi(e_i) \cdot \pi(e_i) = -1$$

(1) $Cl(V)$ 满足的关系.

总的来说, $Cl(V) = \mathrm{span}^{\mathbb{R}}\{1 \text{ (unit of algebra)}, \pi(e_i), \underset{i<j}{\pi(e_i)\pi(e_j)}, \cdots, \pi(e_1) \cdot \pi(e_2) \cdot \cdots \cdot \pi(e_n)\}$ 并且满足关系 (1).

Lemma. π restricts to an embedding of V to $Cl(V)$, i.e. $\pi : V \to \pi(V)$ is a vector space isomorphism.

Remark: 若证明了 lemma, 我们可以将 V 与 $\pi(V)$ 等同起来, 简记 $e_i \sim \pi(e_i)$. 关系 (1) 可写成 $\begin{cases} e_i e_j + e_j e_i = 0, & i \neq j \\ e_i^2 = -1. \end{cases}$.

我们来证明一个比 lemma 更强的结果:

Theorem. The restriction of π to $\wedge V \subset \otimes V$ is an vector space isomorphism onto $Cl(V)$.

proof. 首先给出代数中的一个结果(易证)

proposition: Let $f : V \to \mathcal{A}$ be a linear map, where $\mathcal{A}$ is a associated $\mathbb{R}$-algebra with unit, $f(v) \cdot f(v) + \|v\|^2 \cdot 1 = 0$, $\forall v \in V$, then f extends uniquely to an algebra homomorphism $\hat{f} : \otimes V \to \mathcal{A}$, $\hat{f}$ reduces to an algebra homomorphism $\tilde{f} : Cl(V) \to \mathcal{A}$.

proof of theorem:

Step 1: π is a surjective map.

$i_1 < \cdots < i_k$, $\pi(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pi(\delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} e_{j_1} \otimes \cdots \otimes e_{j_k}) = \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} \pi(e_{j_1}) \cdot \cdots \cdot \pi(e_{j_k}) \overset{(1)}{=} \pi(e_{i_1}) \cdots \pi(e_{i_k})$.

Step 2: π is a injective map.

We construct $H: Cl(V) \rightarrow \Lambda V$ s.t. $H \circ \pi = Id$

- First, recall interior product for $v \in V$. $\iota(v)(u_1 \wedge \cdots \wedge u_k) = \sum_{i=1}^{k} (-1)^{i+1} \langle v, u_i \rangle u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_k$,

  规定: $\iota(v)(1) = 0$.

- We define $F: V \rightarrow End(\Lambda V)$ for $u \in V$, $F(u)(\sigma) = u \wedge \sigma - \iota(u)\sigma$.

  extend $F: \otimes V \rightarrow End(\Lambda V)$ claim $F(u)F(u) + \|u\|^2 Id = 0$.

  proof of claim: take $\sigma = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \Lambda V$

  if $j \notin \{i_1, \cdots, i_k\}$, $F(e_j)F(e_j)\sigma = F(e_j)(e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}) = -\langle e_j, e_j \rangle e_{i_1} \wedge \cdots \wedge e_{i_k} = -\sigma$

  if $j \in \{i_1, \cdots, i_k\}$, $j = i_l$ $F(e_{i_l})F(e_{i_l})\sigma = -F(e_{i_l})((-1)^{l+1} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_l} \wedge \cdots \wedge e_{i_k}) = -\sigma$.

  $\Rightarrow F(v)F(v) + \|v\|^2 Id = 0$. Hence, we get $\tilde{F}: Cl(V) \rightarrow End(\Lambda V)$ ✓

  Define $G: End(\Lambda V) \rightarrow \Lambda V$ $\varphi \mapsto \varphi(1)$. Set $H = G \circ \tilde{F}$ $\sigma = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \Lambda V$

  $H \circ \pi(\sigma) = G \circ \tilde{F} \circ \pi(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} F(e_{j_1}) \cdots F(e_{j_k})(1) = \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k} e_{j_1} \wedge \cdots \wedge e_{j_k} = e_{i_1} \wedge \cdots \wedge e_{i_k} = \sigma$.

  $\Rightarrow H \circ \pi = Id$.

#

Remark: $Cl(V) = span\{1, e_i, \underset{i<j}{e_i e_j}, \cdots, e_1 e_2 \cdots e_n\}$. $\dim Cl(V) = 2^n$.

Also, declaring this basis as being orthonormal we obtain a scalar product on $Cl(V)$.

extending the one on V.

Remark: $Cl^k(V) := span^{\mathbb{R}}\{\underset{i_1 < \cdots < i_k}{e_{i_1} \cdots e_{i_k}}\}$. $Cl^0(V) = \mathbb{R}$.

简单起见, 我们考虑 $V = \mathbb{R}^n$. 记 $Cl(n) = Cl(\mathbb{R}^n)$.

We define $Cl^*(n) = \{\sigma \in Cl(n) : \exists \tau \in Cl(n), \text{ s.t. } \tau\sigma = \sigma\tau = 1\}$ 即 $Cl(n)$ 中全体可逆元构成的群.

Let $v \in \mathbb{R}^n$, $\|v\|^2 \neq 0$, then $v^{-1} = -\|v\|^{-2} v \Rightarrow v \in Cl^*(n)$, i.e. $\mathbb{R}^n \setminus \{0\} \subset Cl^*(n)$.

Theorem. $Cl^*(n)$ is a Lie group of dim 2^n.

proof. We claim: $Cl^*(n)$ is open in $Cl(n)$.

Let $\sigma \in Cl(n)$, set $\exp\sigma := \sum_{k=0}^{\infty} \frac{\sigma^k}{k!}$. $\sum_{k=0}^{\infty} \left|\frac{\sigma^k}{k!}\right| \leq \sum_{k=0}^{\infty} \frac{|\sigma|^k}{k!} = e^{|\sigma|} < \infty$. $(\exp\sigma)^{-1} = \exp(-\sigma)$.

$\Rightarrow \exp: Cl(n) \rightarrow Cl(n)$ well-defined. It is easy to see that exp is smooth and $\exp(Cl(n)) \subset Cl^*(n)$.

claim: $d\exp_0 = Id$.

$\exp(0) = 1$, $Cl(n) \cong \mathbb{R}^{2^n}$. $T_0 Cl(n) \cong \mathbb{R}^{2^n}$. $\frac{d}{dt}\Big|_{t=0} \exp(t\sigma) = \frac{d}{dt}\Big|_{t=0} (1 + t\sigma + \frac{1}{2!}(t\sigma)^2 + \cdots) = \sigma$, i.e. $d\exp_0(\sigma) = \sigma$. ✓

By the inverse mapping theorem, there exists an open neighborhood U of 0, s.t. $\exp|_U$ is a diffemorphism onto $\exp(U)$. $\forall \sigma \in Cl^*(n)$, $L_\sigma : Cl(n) \to Cl(n)$, $L_\sigma(z) = \sigma z$. L_σ is a diffemorphism. $\sigma \cdot \exp(U)$ is an open neighborhood of σ, & $\sigma \exp(U) \subset Cl^*(n)$. $\Rightarrow$ $Cl^*(n)$ is open in $Cl(n)$. Hence, we have $Cl^*(n)$ is a smooth manifold and $Cl^*(n)$ is a Lie group. #

Remark: The Lie algebra of $Cl^*(n)$ is $Cl(n)$.

Def. We define $\mathrm{Spin}(n) = \{u_1 \cdots u_{2k} : u_i \in \mathbb{R}^n,\ \|u_i\| = 1,\ 1 \le i \le 2k,\ k \in \mathbb{N}_+\}$.

Consider the adjoint representation. $Ad : Cl^*(n) \to \mathrm{Aut}(Cl(n))$ $Ad_\sigma \in \mathrm{Aut}(Cl(n))$. $Ad_\sigma(\alpha) = \sigma \alpha \sigma^{-1}$.

the twisted adjoint representation. $\sim : \mathbb{R}^n \to \mathbb{R}^n$ $u \mapsto -u$. $\sigma = v_1 \cdots v_k$ define $\tilde{\sigma} = \tilde{v}_1 \cdots \tilde{v}_k$

$\widetilde{Ad} : Cl^*(n) \to GL(Cl(n))$ $\widetilde{Ad}_\sigma \in GL(Cl(n))$. $\widetilde{Ad}_\sigma(\alpha) := \tilde{\sigma} \alpha \sigma^{-1}$.

We restrict $\widetilde{Ad}$ to $\mathrm{Spin}(n)$. Let $\sigma = u_1 \cdots u_{2k}$ $u_i \in \mathbb{R}^n$, $\|u_i\| = 1$. $Ad_\sigma = Ad_{u_1} \circ \cdots \circ Ad_{u_{2k}}$ and $\widetilde{Ad}_\sigma = \widetilde{Ad}_{u_1} \circ \cdots \circ \widetilde{Ad}_{u_{2k}}$.

Lemma. Let $u \in \mathbb{R}^n$ with $\|u\| = 1$, then $\widetilde{Ad}_u(\mathbb{R}^n) \subset \mathbb{R}^n$ Moreover, $\widetilde{Ad}_u|_{\mathbb{R}^n}$ is the reflection across $u^\perp$

proof. Let $v \in u^\perp$ i.e. $\langle u, v \rangle = 0$. $\widetilde{Ad}_u(v) = \tilde{u} v u^{-1} = v$. $\widetilde{Ad}_u(u) = \tilde{u} u u^{-1} = -u$. #

Set $\xi_0 := \widetilde{Ad}|_{\mathrm{Spin}(n)} = Ad|_{\mathrm{Spin}(n)}$. $\xi_0(\mathrm{Spin}(n)) \subset SO(n)$.

补充: $\mathrm{Aut}(Cl(n))$: 全体代数自同构的集合.

$Ad_\sigma : Cl(n) \to Cl(n)$. 容易看到 Ad_σ 是双射. 还要验证 Ad_σ 是代数同态. i.e $Ad_\sigma(\alpha\beta) = Ad_\sigma(\alpha) \cdot Ad_\sigma(\beta)$

$Ad_\sigma(\alpha\beta) = \sigma(\alpha\beta)\sigma^{-1} = \sigma\alpha\sigma^{-1}\sigma\beta\sigma^{-1} = Ad_\sigma(\alpha) \cdot Ad_\sigma(\beta)$. $\Rightarrow$ $Ad_\sigma \in \mathrm{Aut}(Cl(n))$.

比较之下. $\widetilde{Ad}_\sigma$ 是线性同构, 但未必是代数同构.

②

This lecture, we first continue discussing the mapping ξ_0.

Lemma1. Let V be a finite dimensional inner vector space V. then each $g \in O(V)$ can be written as a product of reflections $g = R_{u_1} \circ \cdots \circ R_{u_k}$, $k \le \dim V$, where R_u is the reflection across $u^\perp$.

proof. 归纳法来证.

- $n=2$. 1° $\det g = 1$.

claim: 设 $\alpha, \beta \in V$, $\alpha \ne \beta$. $|\alpha| = |\beta| = 1$, 则 $\exists\, \eta \in V$. $|\eta| = 1$. $R_\eta(\alpha) = \beta$. 直观想像一下.

proof. 令 $\eta = \frac{\alpha-\beta}{|\alpha-\beta|}$. by the definition of R_η, $\begin{cases} R_\eta(\langle\alpha,\eta\rangle\eta) = -\langle\alpha,\eta\rangle\eta \\ R_\eta(\alpha-\langle\alpha,\eta\rangle\eta) = \alpha-\langle\alpha,\eta\rangle\eta \end{cases}$

$$\Rightarrow R_\eta(\alpha) = \alpha-\langle\alpha,\eta\rangle\eta-\langle\alpha,\eta\rangle\eta = \alpha-2\langle\alpha,\frac{\alpha-\beta}{|\alpha-\beta|}\rangle\frac{\alpha-\beta}{|\alpha-\beta|} = \beta.$$

choose an orthonormal basis of V: $\{\alpha_1,\alpha_2\}$ 设 $g\neq Id$. 不妨设 $g\alpha_1\neq\alpha_1$, 令 $\xi_1 = \frac{\alpha_1-g\alpha_1}{|\alpha_1-g\alpha_1|}$

$R_{\xi_1}(\alpha_1) = g\alpha_1$. $\{R_{\xi_1}(\alpha_1), R_{\xi_1}(\alpha_2)\}$ is a orthonormal basis of V, $\Rightarrow g\alpha_2 = \pm R_{\xi_1}(\alpha_2)$.

若 $g\alpha_2 = R_{\xi_1}(\alpha_2)$ 则 $g = R_{\xi_1}$. 而 $\det g = 1$. $\det R_{\xi_1} = -1$. 矛盾.

$\Rightarrow g\alpha_2 = -R_{\xi_1}(\alpha_2)$ $\Rightarrow g = R_{g\alpha_2}\circ R_{\xi_1}$. Indeed. $R_{g\alpha_2}\circ R_{\xi_1}(\alpha_1) = R_{g\alpha_2}(g\alpha_1) = g\alpha_1$.

$R_{g\alpha_2}\circ R_{\xi_1}(\alpha_2) = R_{g\alpha_2}(-g\alpha_2) = g\alpha_2$.

2° $\det g = -1$.

g 在 V 的一个正交基 e_1, e_2 下的矩阵 A 是正交矩阵. $\det A = -1$.

则 $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, $0\le\theta<2\pi$. 解析几何告诉我们, A 是关于直线 $x\sin\frac{\theta}{2} - y\cos\frac{\theta}{2} = 0$ 的反射.

- 假设对于维数小于 n 命题成立. 现在来看 n 维 $(n\geq 3)$ 的情形.

choose an orthonormal basis of V $\{\eta_1,\cdots,\eta_n\}$.

if $g = Id$. 考虑线性变换 R_{η_1}. $R_{\eta_1}^2 = Id = A$.

if $g\neq Id$. 不妨设 $g\eta_1 = \eta_1$. 由于 $|g\eta_1| = |\eta_1| = 1$. by the claim. $\exists$ reflection B_1. s.t. $B_1(\eta_1) = g\eta_1$.

$\Rightarrow \langle B_1\eta_2,\cdots,B_1\eta_n\rangle = \langle g\eta_2,\cdots,g\eta_n\rangle := U$

考虑线性变换 C. $C(g\eta_i) = B_1\eta_i$ $i=1,2,\cdots,n$. U 是 C 的不变子空间 $C|_U$ 是 U 上的一个正交变换.

由归纳假设知. 存在 U 中至多 $n-1$ 个 reflection $C_2,\cdots,C_s$ $(s\le n)$ s.t. $C|_U = C_2C_3\cdots C_s$. 把 C_j 扩充成 V 上的线性变换 B_j. s.t. $B_j(g\eta_1) = g\eta_1$, $B_j|_U = C_j$. $j=2,\cdots,s$. B_j 是 reflection.

$\Rightarrow g = B_s^{-1}\circ\cdots\circ B_2^{-1}\circ B_1$. Indeed, $B_s^{-1}\circ\cdots\circ B_2^{-1}\circ B_1(\eta_1) = B_s^{-1}\circ\cdots\circ B_2^{-1}(g\eta_1) = g\eta_1$,

$B_s^{-1}\circ\cdots\circ B_2^{-1}\circ B_1(\eta_i) = B_s^{-1}\circ\cdots\circ B_2^{-1}(C(g\eta_i)) = g\eta_i$. #

Theorem 2. ξ_0 is surjective.

proof. take $g\in SO(n)$. by Lemma 1. $g = R_{u_1}\circ\cdots\circ R_{u_{2k}} = \widetilde{Ad}_{u_1}\circ\cdots\circ\widetilde{Ad}_{u_{2k}} = \widetilde{Ad}_{u_1\cdots u_{2k}} = \xi_0(u_1\cdots u_{2k})$. #

Lemma 3 The center of $Cl(n)$ consist of those elements that commute with all $u\in\mathbb{R}^n$. For n even, the center is $\mathbb{R}$, while for n odd, it is $\mathbb{R}\oplus Cl^n(\mathbb{R}^n)$.

proof. to see the mechanism. we could consider the case of $n=3$. 略. #

Theorem4. ξ_0 is 2 to 1. Hence, we have a short exact sequence:

$$1 \to \mathbb{Z}_2 \xrightarrow{i} Spin(n) \xrightarrow{\xi_0} SO(n) \to 1.$$

proof. · $\xi_0^{-1}(Id) = \{1, -1\}$

$\sigma \in \xi_0^{-1}(Id)$. $\Rightarrow \xi_0(\sigma) = Id \Rightarrow \xi_0(\sigma)(u) = u \quad \forall u \in \mathbb{R}^n \Rightarrow \sigma u = u\sigma$, $\forall u \in \mathbb{R}^n \Rightarrow \sigma \in \mathbb{R}$. 而 $|\sigma| = 1$. $\Rightarrow \sigma = \pm 1$.

· $\xi_0^{-1}(g)$ 中有且只有两个元素.

$\sigma \in \xi_0^{-1}(g) \quad \xi_0(\sigma) = g = \xi_0(u_1 \cdots u_{2k})$, by Lemma1. Since ξ_0 is a homomorphism, $\Rightarrow \sigma = \pm u_1 \cdots u_{2k}$. #

Next, we discuss some topological properties of Spin(n).

Lemma5. Let $u \in \mathbb{R}^n$, $\sigma \in Cl(n)$, then $|u\sigma| = |\sigma u| = |u| \cdot |\sigma|$.

Theorem6. Spin(n) is connected $(n \geq 2)$ and compact. Thus, it is an embedded Lie subgroup of $Cl^*(n)$.

proof. Let $a = a_1 \cdots a_{2m} \in Spin(n)$, with a_i in the unit sphere of $\mathbb{R}^n$. Since the sphere is connected, we may connect every a_i by a path $a_i(t)$ to e_1. Hence a can be connected to $e_1 \cdots e_1$ ($2m$ times), which is ± 1. Thus we need to connect 1 and -1. We use the path

$$\gamma(t) = (\cos\tfrac{\pi}{2}t \, e_1 + \sin\tfrac{\pi}{2}t \, e_2)(-\cos\tfrac{\pi}{2}t \, e_1 + \sin\tfrac{\pi}{2}t \, e_2)$$

$$= \cos\pi t + \sin\pi t \, e_1 e_2$$

This path is connected in Spin(n) and satisfies $\gamma(0) = 1$, $\gamma(1) = -1$, and we have shown connectedness of Spin(n) for $n \geq 2$.

· compact? compact = $\underset{?}{\text{closed}}$ + $\underset{\checkmark}{\text{bounded}}$ (by Lemma5)

claim: each $\sigma \in Spin(n)$ can be written as $\sigma = u_1 \cdots u_{2k}$, $2k \leq n$.

denote $T = \xi_0(\sigma)$, then $T \in SO(n)$. $T = \widetilde{Ad}_{u_1} \circ \cdots \circ \widetilde{Ad}_{u_{2k}} = \xi_0(u_1 \cdots u_{2k}) = \xi_0(\sigma)$, $2k \leq n$

ξ_0同态
$\Rightarrow \xi_0(\sigma^{-1} u_1 \cdots u_{2k}) = 1 \Rightarrow \sigma^{-1} u_1 \cdots u_{2k} = \pm 1 \Rightarrow \sigma = \pm u_1 \cdots u_{2k}$.

choose a sequence $\{\sigma^{(k)}\} \subset Spin(n)$.

$$\sigma^{(1)} = u_1^{(1)} u_2^{(1)} \cdots u_{2k}^{(1)} \qquad 2k \leq n$$
$$\sigma^{(2)} = u_1^{(2)} u_2^{(2)} \cdots u_{2k}^{(2)}$$
$$\cdots$$
$$\sigma^{(m)} = u_1^{(m)} u_2^{(m)} \cdots u_{2k}^{(m)}$$
$$\vdots$$
$$\text{逐步选子列} \downarrow$$
$$u_1 \quad u_2 \quad \cdots \quad u_{2k} \in Spin(n).$$

$\Rightarrow$ Spin is closed in $Cl^*(n) \Rightarrow Spin(n)$ is a Lie subgroup of $Cl^*(n)$. #

Theorem 7. ξ_0 is a topological covering map. For $n \geq 3$, ξ_0 is a universal covering map.

proof. · we need that ξ_0 is a local homeomorphism. It suffices to prove that at ± 1. Take disjoint compact neighborhoods $U_\pm$ of ± 1 in $Spin(n)$ with $U_- = -1 \cdot U_+$ and, since ξ_0 is 2 to 1 and surjective, we have $\xi_0 \{ Spin(n) \setminus (U_+ \cup U_-) \} = SO(n) \setminus \xi_0(U_+)$. $\Rightarrow$ $\xi_0(U_+)$ is open. ✓

· Lemma: Every closed loop in $SO(n)$, $n \geq 2$ is homotopic to a loop in $SO(2) \times I_{n-2}$.

proof. $O: [0,1] \longrightarrow SO(n)$ $O(0) = O(1)$, $n \geq 3$. we regularize $O(t)$ to a piecewise C^∞ map. Then the curve $\{O(t)e_n : t \in [0,1]\}$ is of measure 0 in S^{n-1}, so we can choose $\xi \in S^{n-1}$ such that $\pm\xi \notin \{O(t) : t \in [0,1]\}$. Let $O_1 \in SO(n)$ map ξ to e_n. Since $SO(n)$ is connected, our loop is homotopic to the loop $t \mapsto O_1 \circ O(t) = \tilde{O}(t)$, and $\tilde{O}(t) e_n \neq \pm e_n$, $\forall t$. Hence

$$\tilde{O}(t) e_n = e_n \cos\theta(t) + \xi(t) \sin\theta(t)$$

with uniquely determined continuous $\theta(t) \in (0, \pi)$ and $\xi(t) \in S^{n-1}$ orthogonal to e_n.

Denote by $O_2(t,s)$ the rotation by the angle $-s\theta(t)$ in the $e_n \xi(t)$ plane.

将同伦映射写下来:

$$(e_n \ \ \xi(t)) = (e_1, \cdots, e_n) \begin{bmatrix} 0 & \xi^1(t) \\ \vdots & \vdots \\ 0 & \xi^{n-1}(t) \\ 1 & 0 \end{bmatrix} \rightsquigarrow (e_1 \cdots e_n) \begin{bmatrix} \cos\theta & \cdots & & \sin\theta\xi^1(t) \\ & \cos\theta & & \vdots \\ & & \ddots \cos\theta & \sin\theta\xi^{n-1}(t) \\ -\sin\theta\xi^1(t) & \cdots & -\sin\theta\xi^{n-1}(t) & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & \xi^1(t) \\ \vdots & \vdots \\ 0 & \xi^{n-1}(t) \\ 1 & 0 \end{bmatrix}$$

$$= (e_1 \cdots e_n) \begin{bmatrix} \sin\theta\,\xi^1(t) & \cos\theta\,\xi^1(t) \\ \vdots & \vdots \\ \sin\theta\,\xi^{n-1}(t) & \cos\theta\,\xi^{n-1}(t) \\ \cos\theta & -\sin\theta \end{bmatrix}$$

同伦映射对应矩阵.
$$\begin{bmatrix} \cos s\theta(t) & \cdots & 0 & -\sin s\theta(t) \cdot \xi^1(t) \\ \vdots & & \vdots & \\ 0 & \cdots & \cos s\theta(t) & -\sin s\theta(t)\,\xi^{n-1}(t) \\ \sin s\theta(t)\,\xi^1(t) & \cdots & \sin s\theta(t)\,\xi^{n-1}(t) & \cos s\theta(t) \end{bmatrix}$$

$$O_2(t,1) \ (e_n \cos\theta(t) + \xi(t) \sin\theta(t)) = \begin{bmatrix} \cos\theta(t) & \cdots & 0 & -\sin\theta(t)\,\xi^1(t) \\ \vdots & & \vdots & \\ 0 & \cdots & \cos\theta(t) & -\sin\theta(t)\,\xi^{n-1}(t) \\ \sin\theta(t)\,\xi^1(t) & \cdots & \sin\theta(t)\,\xi^{n-1}(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} \xi^1(t)\sin\theta(t) \\ \vdots \\ \xi^{n-1}(t)\sin\theta(t) \\ \cos\theta(t) \end{bmatrix}$$

$$= e_n.$$

$\Rightarrow$ $O_2(t,1) \cdot \tilde{O}(t)$ which leaves e_n fixed, the Lemma is then proved by induction.

- $\mathrm{Spin}(n)$ $n \geq 3$ is simply connected.

Let $t \mapsto x(t) \in \mathrm{Spin}(n)$. $x(0)=x(1)=-1$, be a loop. Then $t \mapsto \xi_0(x(t))$ is a loop in $SO(n)$. $\xi_0(x(0)) = \xi_0(x(1)) = Id$. By the above Lemma, there is a homotopy to a loop in $t \mapsto O(t) \times I_{n-2}$ where $O(t) \in SO(2)$ consists of rotation in $\mathbb{R}^2$ by the angle $2\pi k t$. We can lift the homotopy to one in $\mathrm{Spin}(n)$ connecting the given loop to loop

$$t \mapsto \underbrace{(1,0,\cdots,0)}_{e_1} \cdot \underbrace{(\cos \pi k t, \ -\sin \pi k t, \ 0, \cdots, 0)}_{u(t)}. \qquad k \text{ is even.}$$

验证：上述曲线确实是 $\begin{bmatrix} \cos 2\pi k t & \sin 2\pi k t & \\ -\sin 2\pi k t & \cos 2\pi k t & \\ & & I_{n-2} \end{bmatrix} = O(t)$ 的提升.

证：$\xi_0(e_1 u(t)) = O(t)$

$$\xi_0(e_1 u(t))\, e_1 = e_1 u(t) e_1 (e_1 u(t))^{-1} = e_1 u(t) e_1 u(t)^{-1} (-e_1) = e_1 u(t) e_1 (-u(t)) (-e_1)$$

$$= e_1 (\cos \pi k t\, e_1 - \sin \pi k t\, e_2) e_1 (\cos \pi k t\, e_1 - \sin \pi k t\, e_2) e_1$$

$$= \cos 2\pi k t\, e_1 + \sin 2\pi k t\, e_2$$

$O(t) e_1 = \cos 2\pi k t\, e_1 + \sin 2\pi k t\, e_2 \Rightarrow \xi_0(e_1 u(t)) e_1 = O(t) e_1$

同样可证. $\xi_0(e_1 u(t)) e_2 = O(t) e_2$

$$\xi_0(e_1 u(t)) e_i = e_1 u(t) e_i u(t) e_1 = e_1 (\cos \pi k t\, e_1 - \sin \pi k t\, e_2) e_i (\cos \pi k t\, e_1 - \sin \pi k t\, e_2) e_1$$

$$= (-\cos \pi k t - \sin \pi k t\, e_1 e_2)\, e_i\, (-\cos \pi k t + \sin \pi k t\, e_1 e_2)$$

$$= (-\cos \pi k t - \sin \pi k t\, e_1 e_2)(-\cos \pi k t + \sin \pi k t\, e_1 e_2)\, e_i$$

$$= e_i, \qquad i \geq 3.$$

$O(t) e_i = e_i$. $i \geq 3$ $\Rightarrow$ $\xi_0(e_1 u(t)) e_i = O(t) e_i$, $i \geq 3$.

$\Rightarrow \xi_0(e_1 u(t)) = O(t)$.

When $n \geq 3$ a homotopy to a constant loop is given by

$$(t,s) \longmapsto (1,0,\cdots,0) \cdot (\cos \tfrac{1}{2}\pi s \cos \pi k t, \ -\cos \tfrac{1}{2}\pi s \sin \pi k t, \ \sin \tfrac{1}{2}\pi s, \ 0, \ \cdots, 0).$$

where $t \in [0,1]$. $s \in [0,1]$. Hence $\mathrm{Spin}(n)$ is simply connected. #

Remark: $n=2$. $\mathrm{Spin}(2)$ is diffeomorphic to S^1. $SO(2) \cong S^1$.

Hence, the theorem is false for $n=2$.

Theorem. The Lie algebra of $\mathrm{Spin}(n)$ is $\mathfrak{spin}(n) = Cl^2(n) = \mathrm{span}^{\mathbb{R}}\{e_ie_j : i<j\}$.

proof. By topology theory. $\dim \mathrm{Spin}(n) = \dim SO(n) = \frac{n(n-1)}{2}$, $\Rightarrow$ $\dim \mathfrak{spin}(n) = \frac{n(n-1)}{2}$

it suffices to prove that $Cl^2(n) \subset \mathfrak{spin}(n)$.

construct a C^∞ curve. $\gamma(t) = (\cos\frac{t}{2} e_1 + \sin\frac{t}{2} e_2)(-\cos\frac{t}{2} e_1 + \sin\frac{t}{2} e_2)$

$$= \cos t + \sin t\, e_1e_2$$

$\gamma(0) = 1$. $\gamma'(0) = e_1e_2$ ✓

#

Finally, we discuss some smooth properties of ξ_0.

Theorem: ξ_0 is smooth. $(d\xi_0)_1 : \mathfrak{spin}(n) \to \mathfrak{so}(n)$, where $(v\wedge w)(u) = \langle v, u\rangle w - \langle w, u\rangle v$.

$(d\xi_0)_1(e_ie_j) = 2e_i\wedge e_j$

proof.

$$\begin{tikzcd} \mathrm{Ad}: & Cl^*(n) \arrow[r] & \mathrm{Aut}(Cl(n)) & \\ & \cup & \cup \arrow[r] & \text{embedded} \\ \xi_0 & \mathrm{Spin}(n) \arrow[r] & SO(n) & \end{tikzcd}$$

$\Rightarrow$ ξ_0 is smooth.

$(d\xi_0)_1(e_ie_j) = \frac{d}{dt}\Big|_{t=0} \xi_0\circ\gamma(t)$ $\qquad$ $\gamma(t) = \cos t + \sin t\, e_ie_j$ $\qquad$ $\gamma(t)\gamma^{-1}(t) = 1$

$\Rightarrow (d\xi_0)_1(e_ie_j)(u) = \frac{d}{dt}\Big|_{t=0} \xi_0(\gamma(t))(u) = \frac{d}{dt}\Big|_{t=0} \gamma(t)u\gamma^{-1}(t) = \gamma'(0)u\gamma^{-1}(0) + \gamma(0)u(\gamma^{-1}(t))'\Big|_{t=0} \overset{\downarrow}{=} e_ie_ju - ue_ie_j$

$\Rightarrow (d\xi_0)_1(e_ie_j) = 2e_i\wedge e_j$. $\Rightarrow$ $\frac{1}{4}(d\xi_0)_1([u,v]) = u\wedge v$.

#

Theorem ξ_0 is a local diffeomorphism and smooth 2 to 1 covering.

proof. ξ_0 : Lie group homomorphism $\Rightarrow$ ξ_0 : constant rank & $(d\xi_0)_1$ is nonsingular.

$\Rightarrow$ ξ_0 is a local diffeomorphism.^{*1} + ξ_0 is a topological covering map.^{*2}

$\overset{*1}{\underset{*2}{\Rightarrow}}$ ξ_0 is a smooth covering map.

#

③

• Clifford Module

Def. Let k is a field, K also a field, s.t. $k\subset K$. Let V be a k-vector space, q k-quadratic form on V.

Let W be a K-vector space. A K-representation of $Cl(V,q)$ on W is a k-algebra homomorphism,

$$\rho : Cl(V,q) \to \mathrm{End}_K W$$

then W is called a Clifford module.

SPIN GEOMETRY 3

LITTLE PRINCE

1. Clifford module

Definition 1.1. *Let k is a field, K also a field, s.t.$k \subset K$, Let V be a k-vector space, q k-quadratic form on V. Let W be a K-vector space. a K-representation of $Cl(V, q)$ on W is a k-algebra homomorphism,*

$$\rho : Cl(V, q) \to End_K(W). \tag{1.1}$$

W is called a Clifford module.

want: establish a algebra isomorphism between $Cl^{\mathbb{C}}(V)$ and $End_{\mathbb{C}}(S)$.

¶ what is $Cl^{\mathbb{C}}(V)$?

Let V be a $n = 2m$ dimensional real inner vector space, assume that $e_1, \cdots, e_n$ is a basis of V. Let $V^{\mathbb{C}}$ be the complexification of V. We now extend the scalar product $\langle\cdot,\cdot\rangle$ from V to $V^{\mathbb{C}}$ as a Hermitian product, i.e.

$$\langle\sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j\rangle = \sum_{i=1}^{n} \alpha_i \bar{\beta}_i, \tag{1.2}$$

for $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \mathbb{C}$. Set

$$W = span^{\mathbb{C}}\{\eta_1, \cdots, \eta_m\}, \tag{1.3}$$

where $\eta_j = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$. Set

$$\bar{W} = span^{\mathbb{C}}\{\bar{\eta_1}, \cdots, \bar{\eta_m}\}, \tag{1.4}$$

then we have

$$V^{\mathbb{C}} = W \oplus \bar{W}. \tag{1.5}$$

Set

$$\bigotimes V^{\mathbb{C}} = \bigoplus_{k\geq 0} \otimes^k V^{\mathbb{C}}, \tag{1.6}$$

where $\otimes^k V^{\mathbb{C}} = \{k$ order complex-valued, real linear function on $V\}$. Set

$$\mathcal{H}^{\mathbb{C}} = \{\sum_{i=1}^{m} w_i \otimes (v_i \otimes v_i + \|v_i\|) \otimes z_i : w_i, z_i \in \bigotimes V^{\mathbb{C}}, v_i \in V^{\mathbb{C}}, m \in N_+\}. \tag{1.7}$$

We define

$$Cl^{\mathbb{C}}(V) = \bigotimes V^{\mathbb{C}}/\mathcal{H}^{\mathbb{C}}. \tag{1.8}$$

We have

$$\pi(v \otimes v + \|v\|) = 0, \text{ for all } v \in V^{\mathbb{C}} \Leftrightarrow e_i e_j + e_j e_i = 0, i < j, \text{ and } e_i^2 = -1.$$

Definition 1.2. *The spinor space S is defined as the exterior algebra $\bigwedge W$ of W.*

$v = w + w' \in W \oplus \bar{W}$, $s \in S = \bigwedge W$, define

$$\rho(w)s := \sqrt{2}\epsilon(w)s, \ \epsilon: \text{ exterior product };$$
$$\rho(w')s := -\sqrt{2}\iota(w')s, \ \iota: \text{ interior product }.$$

$\rho : V^{\mathbb{C}} \to End_{\mathbb{C}}(\bigwedge W)$ is a linear map, then ρ extends uniquely to an algebra homomorphism $\tilde{\rho} : \bigotimes V^{\mathbb{C}} \to End_{\mathbb{C}}(\bigwedge W)$.
Claim: $\epsilon(\eta_j)\iota(\bar{\eta}_j) + \iota(\bar{\eta}_j)\epsilon(\eta_j)) = Id$, and $\epsilon(\eta_j)\iota(\bar{\eta}_l) + \iota(\bar{\eta}_l)\epsilon(\eta_j) = 0$, for $j \neq l$.
It is easy to verify that

$$\rho(e_i)\rho(e_i) = Id, \ \rho(e_i)\rho(e_j) + \rho(e_j)\rho(e_i) = 0.$$

$\Rightarrow \tilde{\rho}$ reduces to an algebra homomorphism $\rho : Cl^{\mathbb{C}}(V) \to End_{\mathbb{C}}(\bigwedge W)$.
Note that $\rho : Cl^{\mathbb{C}}(V) \to End_{\mathbb{C}}(\bigwedge W)$ is surjective. Indeed, we have
$\rho(\eta_{i_1} \cdots \eta_{i_k} \bar{\eta}_1 \cdots \bar{\eta}_m \eta_1 \cdots \eta_m \bar{\eta}_{i_1} \cdots \bar{\eta}_{i_k})(\eta_{i_1} \cdots \eta_{i_k}) = const \cdot \eta_{i_1} \cdots \eta_{i_k}$,
and $\rho(\eta_{i_1} \cdots \eta_{i_k} \bar{\eta}_1 \cdots \bar{\eta}_m \eta_1 \cdots \eta_m \bar{\eta}_{i_1} \cdots \bar{\eta}_{i_k})$ maps all other basis vectors of $\bigwedge W$ to zero.
We know $Cl^{\mathbb{C}}(V)$ and $End_{\mathbb{C}}(\bigwedge W)$ have the same dimension.Hence, we have the following theorem.

Theorem 1.1. *If $n = dim_{\mathbb{R}} V$ is even, $Cl^{\mathbb{C}}(V)$ is isomorphism to the algebra of complex linear endomorphisms of the spinor space S or S_n.*

For the sequel, we need to choose an orientation of V, i.e. select an orthonormal basis $\{e_1, \cdots, e_n\}$ of V being positive.

Definition 1.3. *Let $\{e_1, \cdots, e_n\}$ be a positive orthonormal basis of V. The chirality operator is*

$$\Gamma = i^m e_1 \cdots e_n \in Cl^{\mathbb{C}}(V) \tag{1.9}$$

with $m = \frac{n}{2}$ for even n, $m = \frac{n+1}{2}$ for odd n.

Remark 1.2. *It is easy to check that Γ is independent of the chosen positive orthonormal basis.*

- $\eta_j \bar{\eta}_j - \bar{\eta}_j \eta_j = 2ie_{2j-1}e_{2j}$, we have $\Gamma = 2^{-m}(\eta_1\bar{\eta}_1 - \bar{\eta}_1\eta_1) \cdots (\eta_m\bar{\eta}_m - \bar{\eta}_m\eta_m)$.
- $\rho(\Gamma) = (-1)^m(\epsilon(\eta_1)\iota(\bar{\eta}_1) - \iota(\bar{\eta}_1)\epsilon(\eta_1)) \cdots (\epsilon(\eta_m)\iota(\bar{\eta}_m) - \iota(\bar{\eta}_m)\epsilon(\eta_m))$;
  $\rho(\Gamma) = (-1)^k$ on $\bigwedge^k W$.
  Set S^+: elements of even degree; S^-: elements of odd degree.

Claim:
1. $Spin(n)$ leaves the spaces S^+ and S^- invariant;
2. the representation $\rho : Spin(n) \to End_{\mathbb{C}}(S^+)$ is irreducible.
3. multiplication by a vector $v \in V$, exchanges S^+ and S^-.

Proof. To see the mechanism, let us just consider some case.
1. assume that $i_1 > 1$,

$$\begin{aligned}
(1.10)\rho(e_1e_2)(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) &= (\epsilon(\eta_1)-\iota(\bar{\eta_1}))i(\epsilon(\eta_1)+\iota(\bar{\eta_1}))(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) \\
(1.11) \qquad &= -i\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}
\end{aligned}$$

$$\begin{aligned}
(1.12)\rho(e_1e_2)(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) &= (\epsilon(\eta_1)-\iota(\bar{\eta_1}))i(\epsilon(\eta_1)+\iota(\bar{\eta_1}))(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) \\
(1.13) \qquad &= i\eta_1\wedge\cdots\wedge\eta_{i_{2k}}
\end{aligned}$$

2. assume that $i_1 \geq 3$,

$$\begin{aligned}
(1.14)\rho(e_2e_3)(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) &= (\epsilon(\eta_1)+\iota(\bar{\eta_1}))i(\epsilon(\eta_2)-\iota(\bar{\eta_2}))(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) \\
(1.15) \qquad &= i\eta_1\wedge\eta_2\wedge\cdots\wedge\eta_{i_{2k}}
\end{aligned}$$

$$\begin{aligned}
\rho(e_2e_3)(\eta_1\wedge\eta_2\wedge\eta_{i_3}\wedge\cdots\wedge\eta_{i_{2k}}) \ (1.16) &= (\epsilon(\eta_1)+\iota(\bar{\eta_1}))i(\epsilon(\eta_2)-\iota(\bar{\eta_2}))(\eta_1\wedge\eta_2\wedge\eta_{i_3}\wedge\cdots\wedge\eta_{i_{2k}}) \\
(1.17) \qquad &= i\eta_{i_3}\wedge\cdots\wedge\eta_{i_{2k}}
\end{aligned}$$

3. assume that $i_1 = 1$,

$$\rho(e_1)(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) = -\eta_{i_2}\wedge\cdots\wedge\eta_{i_{2k}}, \tag{1.18}$$

assume that $i_1 > 1$,

$$\rho(e_1)(\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}) = \eta_1\wedge\eta_{i_1}\wedge\cdots\wedge\eta_{i_{2k}}. \tag{1.19}$$

□

Definition 1.4. *The above representation ρ of $Spin(V)$ on the spinor space S is called the spinor representation, and the representations on S^+ and S^- are called half spinor representations.*

Odd dimensioanl?

Proposition 1.3. *If $dim_{\mathbb{R}}V = n = 2m+1$, then $Cl^{\mathbb{C}}(V) \simeq End_{\mathbb{C}}(S_{2m}) \oplus End_{\mathbb{C}}(S_{2m})$.*

Definition 1.5. *If $dim_{\mathbb{R}}V = n = 2m+1$, the complex spinor representation is defined to be the projection onto the first component of the corresponding isomorphism.*

The scalar product $\langle\cdot,\cdot\rangle$ from V to $V^{\mathbb{C}}$ extends to $\bigwedge V$ by letting the monomials $e_{i_1}\wedge\cdots\wedge e_{i_k}, 1 \leq i_1 < \cdots i_k \leq n$, constitute an orthonormal basis.
Claim: $\langle\rho(e_j)s, \rho(e_j)s'\rangle = \langle s, s'\rangle$, $\forall s, s' \in \bigwedge W$. Of course, this then holds more generally for every $v \in V$ with $\|v\| = 1$, and also for products $v_1\cdots v_k$ with $\|v_j\| = 1$ for $j = 1, \cdots, k$. This implies

Corollary 1.4. *The induced representation of $Spin(V)$ on $End_{\mathbb{C}}(S)$ preserves the Hermitian product.*

Corollary 1.5. $\langle\rho(v)s, s'\rangle = -\langle s, \rho(v)s'\rangle$, $\forall s, s' \in \bigwedge W, v \in V$.

2. CONNECTIONS

Definition 2.1. *Given a Lie group G, a G-principal fibre bundle is a triple (P, π, M) such that (i) $\pi : P \to M$ is a smooth map between finite dimensional smooth manifolds.*
(ii) G acts smoothly and freely on P from the right, i.e. the action $P \times G \to P$ satisfies $pg = p$ if and only if $g = e \in G$.
(iii) For every point $x \in M$, there exists an open neighborhood $U \subset M$ and a diffeomorphism, called local trivialisation, $\psi : U \times G \to \pi^{-1}(U)$, s.t.
$\pi \circ \psi(y, g) = y, \forall (y, g) \in U \times G$, and $\psi(y, g)h = \psi(y, gh), \forall h \in G$.

Definition 2.2. *Let $\pi : P \to M$ be a principal bundle, a connection 1-form ω is an element in $\Omega^1(P) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is Lie algebra of G, s.t.*
(1) $R_g^*(\omega) = Ad(g^{-1}) \circ \omega$ for any $g \in G$.
(2)

$$\omega_p\left(\frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tv)\right) = v, \tag{2.1}$$

for any $v \in \mathfrak{g}, p \in P$.

Definition 2.3. *Let $\pi : P \to M$ be a principal bundle, a connection 1-form ω is a distribution of n-dimensional vector spaces $p \mapsto H_p \subset T_pP$, the horizontal spaces, such that (i) $T_pP = V_p \oplus H_p$, where $V_p = ker(\pi_{*p})$, the vertical space V_p.*
(ii) it is G-invariant, i.e. $H_{pg} = (R_g)_{*p}(H_p)$.

SPIN GEOMETRY 4

LITTLE PRINCE

1. PRELIMINARY

Theorem 1.1. *Let $\pi : P \to M$ be a G-principal bundle, let ω be a connection 1-form, let $\rho : G \to GL(V)$ be a representation of G, where V is a vector space, then we have a induced connection on the associated vector bundle $P \times_\rho V$.*

Proof. ¶ what is $P \times_\rho V$? Given a representation $\rho : G \to GL(V)$ of G, we can define a action of G on V:

$$G \times V \to V, (g, v) := \rho(g)(v). \tag{1.1}$$

We construct the associated vector bundle $E \to M$ with fiber V as follows:
We have a free action of G on $P \times V$ from the right:

$$\begin{gathered} P \times V \times G \to P \times V \\ (p, v) \cdot g \to (p \cdot g, \rho(g^{-1})v). \end{gathered}$$

If we divide out this G-action, i.e. identify (p, v) and $(p, v) \cdot g$, we obtain the quotient space $P \times_\rho V$. Set $\tilde{\pi} : P \times_\rho V \to M$, s.t. $\tilde{\pi}([p, v]) = \pi(p)$. Given $p \in P$, denote $x = \pi(p)$, then p induces a natural map $\phi_p : V \to \tilde{\pi}^{-1}(p)$:

$$\phi_p(v) = [(p, v)]. \tag{1.2}$$

We can verify that $P \times_\rho V$ is a vector bundle.
¶ Now, in order to define the connection on $P \times_\rho V$, we need the following proposition.

Proposition 1.2. *Let H be a connection on P, $x \in M$, let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve, set $\gamma(0) = x$, then for every $p \in \pi^{-1}(x)$, there exists a smooth curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to P$ uniquely, s.t. $\tilde{\gamma}(0) = p$, $\pi(\tilde{\gamma}(t)) = \gamma(t)$, and $\tilde{\gamma}'(t) \in H(\tilde{\gamma}(t))$.*

¶ assume that s is a local section of $P \times_\rho V$, $x \in M, X \in T_xM$, take a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = x, \gamma'(0) = X$. By the above proposition, for every $p \in \pi^{-1}(x)$, there exists a smooth curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to P$, s.t. $\tilde{\gamma}(0) = p$, $\pi(\tilde{\gamma}(t)) = \gamma(t)$, and $\tilde{\gamma}'(t) \in H(\tilde{\gamma}(t))$.

$$s(\gamma(t)) = [(\tilde{\gamma}(t), v(t))], \tag{1.3}$$

where $v(t)$ is a smooth curve in V. Set

$$\nabla_X s = [(p, v'(0))]. \tag{1.4}$$

□

2. Clifford bundle

Let $\pi : E \to M$ be a oriented vector bundle, let $P = P_{SO}(E)$ be the frame bundle over E with fiber $SO(n)$. $SO(n)$ acts on $Cl(\mathbb{R}^n)$ simply by extending the action of $SO(n)$ on $\mathbb{R}^n$. Thus, P induces the Clifford bundle:

$$Cl(P) = P \times_{SO(n)} Cl(\mathbb{R}^n). \tag{2.1}$$

Lemma 2.1. *For smooth sections σ, τ of $Cl(P)$ we have*

$$\nabla(\sigma\tau) = \nabla(\sigma)\tau + \sigma\nabla(\tau). \tag{2.2}$$

Proof. assume that $x \in M, X \in T_xM$, take a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = x, \gamma'(0) = X$. For every $p \in \pi^{-1}(x)$, there exists a smooth curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to P$, s.t. $\tilde{\gamma}(0) = p$, $\pi(\tilde{\gamma}(t)) = \gamma(t)$, and $\tilde{\gamma}'(t) \in H(\tilde{\gamma}(t))$.

$$\sigma(\gamma(t)) = [(\tilde{\gamma}(t), v(t))], \tau(\gamma(t)) = [(\tilde{\gamma}(t), w(t))], \tag{2.3}$$

where $v(t), w(t)$ is a smooth curve in V.

$$\begin{aligned} \nabla_X(\sigma\tau) &= [p, \left.\frac{d}{dt}\right|_{t=0} v(t)w(t)] && (2.4)\\ &= [p, v'(0)w(0) + v(0)w'(0)] && (2.5)\\ &= [p, v'(0)][p, w(0)] + [p, v(0)][p, w'(0)] && (2.6)\\ &= \nabla_X\sigma \cdot \tau(x) + \sigma(x) \cdot \nabla_X\tau. && (2.7) \end{aligned}$$

□

3. spinor bundle

Definition 3.1. *Let $\pi : E \to M$ be a oriented Riemannian vector bundle with rank n, a spin structure on E is a Spin(n)-bundle $P_{Spin(n)}$ over M, and there exists a map $\xi : P_{Spin(n)} \to P_{SO}(E)$, such that $\xi((P_{Spin(n)})_x) = P_{SO}(E)_x$ and $\xi(\sigma g) = \xi(\sigma)\xi_0(g)$, where σ is a smooth section of $\sigma \in P_{Spin(n)}$ and $g \in Spin(n)$.*

Theorem 3.1. *Let ω be a connection 1-form on $P_{SO}(E)$, we define*

$$\omega_{Spin(n)} := (d\xi_0)_1^{-1} \circ (\xi^*\omega). \tag{3.1}$$

Then $\omega_{Spin(n)}$ is a connection 1-form on $P_{Spin(n)}$.

Remark 3.2. *$\pi : P_{SO}(E) \to M$, $(\pi^{-1}(U); x^i, A^b_a)$ is a local coordinate of $P_{SO}(E)$, we define a form on $P_{SO}(E)$. $\nabla e_a = \omega^b_a e_b$.*

$$\theta^b_a = (A^{-1})^b_c(dA^c_a + \pi^*(\omega^c_d)A^d_a). \tag{3.2}$$

assume that $s = e \cdot A$, then we have

$$\nabla s_a = (A^{-1})^b_c(dA^c_a + \omega^c_d A^d_a)s_b. \tag{3.3}$$

Proof. ¶ $R_g^*(\omega_{Spin(n)}) = Ad(g^{-1}) \circ \omega_{Spin(n)}$ for any $g \in Spin(n)$.
assume that $\eta \in T_pP, p \in P$,

$$\begin{align}
R_g^*(\omega_{Spin(n)})(\eta) &= \omega_{Spin(n)}((R_g)_*(\eta)) \tag{3.4}\\
&= (d\xi_0)_1^{-1} \circ (\xi^*\omega)((R_g)_*(\eta)) \tag{3.5}\\
&= (d\xi_0)_1^{-1}(\omega(\xi \circ R_g)_*(\eta)) \tag{3.6}\\
&= (d\xi_0)_1^{-1}(\omega(R_{\xi_0(g)} \circ \xi)_*(\eta)) \tag{3.7}\\
&= (d\xi_0)_1^{-1}(R_{\xi_0(g)}^*\omega(\xi_*(\eta)) \tag{3.8}\\
&= (d\xi_0)_1^{-1}(Ad(\xi_0(g)^{-1})(\omega(\xi_*(\eta))) \tag{3.9}\\
&= (d(ad(\xi_0(g)) \circ \xi_0))^{-1}(\omega(\xi_*(\eta))) \tag{3.10}\\
&= (d(\xi_0 \circ ad(g)))^{-1}(\omega(\xi_*(\eta))) \ \xi_0 \text{ is a homomorphism} \tag{3.11}\\
&= Ad(g^{-1})(d\xi_0)_1^{-1} \circ (\xi^*\omega)(\eta)) \tag{3.12}\\
&= Ad(g^{-1}) \circ \omega_{Spin(n)}(\eta). \tag{3.13}
\end{align}$$

¶ $\omega_{Spin(n)p}\left(\frac{d}{dt}\Big|_{t=0} p \cdot \exp(tv)\right) = v$, where $v \in spin(n)$.

$$\begin{align}
\omega_{Spin(n)p}\left(\frac{d}{dt}\Big|_{t=0} p \cdot \exp(tv)\right) &= (d\xi_0)_1^{-1} \circ (\xi^*\omega)\left(\frac{d}{dt}\Big|_{t=0} p \cdot \exp(tv)\right) \tag{3.14}\\
&= (d\xi_0)_1^{-1} \circ \omega\left(\frac{d}{dt}\Big|_{t=0} \xi(p \cdot \exp(tv))\right) \tag{3.15}\\
&= (d\xi_0)_1^{-1} \circ \omega\left(\frac{d}{dt}\Big|_{t=0} \xi(p) \cdot \xi_0(\exp(tv))\right) \tag{3.16}\\
&= (d\xi_0)_1^{-1} \circ \omega\left(\frac{d}{dt}\Big|_{t=0} \xi(p) \cdot (\exp t(d\xi_0)_1(v))\right) \tag{3.17}\\
&= (d\xi_0)_1^{-1} \circ (d\xi_0)_1(v) \tag{3.18}\\
&= v. \tag{3.19}
\end{align}$$

□

Now, we assume that W is a Clifford module with $\rho : Cl(n) \to End(W)$, then we have the following restriction map

$$\rho : Spin(n) \to GL(W) \tag{3.20}$$

Thus, we have $F = P_{Spin(n)} \times_\rho W$. By theorem 1.1, we have the induced connection on F. We need to calculate its concrete form.

Let $x \in M, X \in T_xM$, let $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x, \gamma'(0) = X$. Take $p \in \pi^{-1}(x)$, there exists a curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to P$ such that

$$\tilde{\gamma}(0) = p, \pi(\tilde{\gamma}(t)) = \gamma(t), \tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}. \tag{3.21}$$

Let $[(\sigma, w)]$ be a local section of F, where σ is a local section of $P_{Spin(n)}$ such that $\sigma(x) = p$. Thus, there exists a curve $g : (-\epsilon, \epsilon) \to Spin(n)$ such that

$$g(0) = e, \sigma(\gamma(t)) = \tilde{\gamma}(t) \cdot g(t). \tag{3.22}$$

We calculate

$$\begin{aligned}
\nabla_X[(\sigma, w)] &= \nabla_X[\tilde{\gamma}(t), \rho(g(t))(w)] \tag{3.23}\\
&= [(p, \rho(g'(0))(w))] \tag{3.24}\\
&= [(p, \rho(\omega_{spin(n)}\left(\frac{d}{dt}\Big|_{t=0} p \cdot \exp(tg'(0))\right)(w))] \tag{3.25}\\
&= [(p, \rho(\omega_{spin(n)}\left(\frac{d}{dt}\Big|_{t=0} p \cdot g(t)\right)(w))] \tag{3.26}\\
&= [(p, \rho(\omega_{spin(n)}\left(\frac{d}{dt}\Big|_{t=0} \tilde{\gamma}(t) \cdot g(t) - \tilde{\gamma}'(0)\right)(w))] \tag{3.27}\\
&= [(p, \rho((d\xi_0)_1^{-1} \circ (\xi^*\omega)(\sigma_*(X)))(w))] \tag{3.28}\\
&= [(p, \rho((d\xi_0)_1^{-1} \circ (s^*\omega)(X))(w))], set\ \ s = \xi \circ \sigma \tag{3.29}\\
&= [(p, \rho((d\xi_0)_1^{-1}(-\sum_{i<j} \omega^i_j(X)\varepsilon_i \wedge \varepsilon_j(w))))] \tag{3.30}\\
&= \frac{1}{2}\sum_{i<j} \omega^j_i(X)[(p, \rho(\varepsilon_i)\rho(\varepsilon_j)w)] \tag{3.31}\\
&= \frac{1}{2}\sum_{i<j} \omega^j_i(X)[(p, \varepsilon_i \cdot \varepsilon_j \cdot w)] \tag{3.32}\\
&= \frac{1}{2}\sum_{i<j} \omega^j_i(X)[(p, \varepsilon_i)] \cdot [(p, \varepsilon_j)] \cdot [(p, w)] \tag{3.33}\\
&= \frac{1}{2}\sum_{i<j} \omega^j_i(X)e_i e_j[(p, w)] \tag{3.34}
\end{aligned}$$

where (ω^i_j) is the connection form of E w.r.t s, $\{e_i\}$ is a local orthonormal frame field of E. Here, we used $E \cong P_{SO}(E) \times_{\rho_0} \mathbb{R}^n \cong P_{Spin(n)} \times_{\xi_0} \mathbb{R}^n$. Indeed, assume that $(e_1, \cdots, e_n)$ is a local frame of E, we have

$$v^1 e_1 + \cdots v^n e_n \leftrightarrow [(e_1, \cdots, e_n)A, A^{-1}(v^1, \cdots, v^n)^T] \leftrightarrow [\sigma g, \xi_0(g^{-1})(v^1, \cdots, v^n)^T]$$

where $\xi(\sigma) = (e_1, \cdots, e_n)$ and $\xi_0(g) = A$. Hence, we obtain that

$$\nabla\varphi = \frac{1}{2}\sum_{i<j} \omega^j_i e_i e_j \varphi, \tag{3.35}$$

for any local section φ of F.
Now we **claim**

$$\nabla(s \cdot \varphi) = \nabla s \cdot \varphi + s \cdot \nabla\varphi, \tag{3.36}$$

where s is a smooth section of $P_{Spin(n)} \times_{\xi_0} \mathbb{R}^n$ and φ is a smooth section of F.

We can also calculate **the curvature operator** on F:

$$R(X,Y)\varphi = \frac{1}{2}\sum_{i<j}\langle R(X,Y)e_i, e_j\rangle e_i e_j \varphi. \tag{3.37}$$

for any local vector fields X, Y on M and local section φ of F.

Proof. Let $x \in M, X, Y \in T_xM$, assume that $[\overline{X}, \overline{Y}](x) = 0$ where $\overline{X}, \overline{Y}$ are local smooth vector field with $\overline{X}(x) = X, \overline{Y}(y) = Y$, and $(e_1, \cdots, e_n)$ is a local frame of E with $\nabla e_i = 0$ for any $i = 1, \cdots, n$. $\square$

SPIN GEOMETRY 5

LITTLE PRINCE

1. The Dirac operator

Definition 1.1. *Let (M, g) be a oriented Riemannian manifold, we say that (M, g) is a spin manifold if there exists a spin structure $P \to M$ on TM. We also say that $P \to M$ is a spin structure on M.*

Definition 1.2. *Let $P \to M$ be a spin structure on the oriented Riemannian manifold M, with Levi-Civita connection and even dimension. $\rho : Spin(n) \to GL_{\mathbb{C}}(S^n)$. Consider the spinor bundle*

$$\mathcal{S}_n = P \times_\rho S_n. \tag{1.1}$$

The Dirac operator $\not{\partial}$ operates on sections φ of the spinor bundle $\mathcal{S}_n$ via

$$\not{\partial}\varphi(p) = e_i \nabla_{e_i} \varphi \tag{1.2}$$

where $\{e_i\}$ is an orthonormal basis of T_pM and here we used the vector isomorphism $TM \cong P_{Spin(n)} \times_{\xi_0} \mathbb{R}^n$.

Remark 1.1. *Odd dimensional?*
If $dim_{\mathbb{R}} V = n = 2m + 1$, then $Cl^{\mathbb{C}}(V) \simeq End_{\mathbb{C}}(S_{2m}) \oplus End_{\mathbb{C}}(S_{2m})$, the complex spinor representation is defined to be the projection onto the first component of the corresponding isomorphism.

Lemma 1.2. *The Dirac operator $\not{\partial}$ do not depend on the choice of an orthonormal frame e_i.*

Proof. Any other such frame $\{f_j\}$ can be obtained as $e_k = a_k^l f_l$ for some $(a_k^l) \in SO(n)$. Then

$$\begin{aligned}
e_k \nabla_{e_k}[(\sigma, w)] &= [(p, \epsilon_k)] \frac{1}{4} \omega_i^j(e_k) e_i e_j [(p, w)] && (1.3)\\
&= \frac{1}{4} a_i^p \tilde{\omega}_p^q a_j^q (a_k^l f_l) [(p, \epsilon_k \epsilon_i \epsilon_j w)] && (1.4)\\
&= \frac{1}{4} \tilde{\omega}_p^q (f_l) [(p, a_k^l \epsilon_k)] [(p, a_i^p \epsilon_i)] [(p, a_j^q \epsilon_j)] [(p, w)] && (1.5)\\
&= f_l \frac{1}{4} \tilde{\omega}_p^q (f_l) f_p f_q [(p, w)] && (1.6)\\
&= f_l \nabla_{f_l} [(\sigma, w)] && (1.7)
\end{aligned}$$

□

Lemma 1.3. *Let M be even dimensional, and let $\mathcal{S}_n^{\pm}$ be the half spinor bundles for a spin structure on M. Then the Dirac operator $\partial\!\!\!/$ maps $\Gamma(\mathcal{S}_n^{\pm})$ to $\Gamma(\mathcal{S}_n^{\mp})$.*

Recall that on the spinor space S_n, we have a pointwise Hermitian product $\langle\cdot,\cdot\rangle$(invariant under $Spin(n)$). We suppose now that M is compact. We may then form the associated L^2 product

$$(\varphi_1,\varphi_2) := \int_M \langle\varphi_1(x),\varphi_2(x)\rangle dv. \tag{1.8}$$

where φ_1,φ_2 are smooth sections on $\mathcal{S}_n$.

Lemma 1.4. ∇ *is compatible with the* $\langle\cdot,\cdot\rangle$*:*

$$\nabla_X\langle\varphi_1,\varphi_2\rangle = \langle\nabla_X\varphi_1,\varphi_2\rangle + \langle\varphi_1,\nabla_X\varphi_2\rangle. \tag{1.9}$$

for any smooth vector field X on M and smooth section φ_1,φ_2 on $\mathcal{S}_n$.

Proof. Let $x\in M$ and $X\in T_xM$, take a smooth curve $\gamma:(-\epsilon,\epsilon)\to M$ such that $\gamma(0)=x,\gamma'(0)=X$. For every $p\in\pi^{-1}(x)$, there exists a smooth curve $\tilde\gamma:(-\epsilon,\epsilon)\to P$, s.t. $\tilde\gamma(0)=p$, $\pi(\tilde\gamma(t))=\gamma(t)$, and $\tilde\gamma'(t)\in H(\tilde\gamma(t))$.

$$\varphi_1(\gamma(t)) = [(\tilde\gamma(t),v(t))], \varphi_2(\gamma(t)) = [(\tilde\gamma(t),w(t))], \tag{1.10}$$

where $v(t),w(t)$ is a smooth curve in V.

$$\begin{aligned}
\nabla_X\langle\varphi_1,\varphi_2\rangle &= \left.\frac{d}{dt}\right|_{t=0}\langle v(t),w(t)\rangle && (1.11)\\
&= \langle\nabla_X\varphi_1,\varphi_2\rangle + \langle\varphi_1,\nabla_X\varphi_2\rangle. && (1.12)
\end{aligned}$$

□

Lemma 1.5. *Let M be a compact Riemannian manifold with a spin structure. Then the corresponding Dirac operator $\partial\!\!\!/$ is formally selfadjoint, i.e.*

$$(\partial\!\!\!/\varphi_1,\varphi_2) = (\varphi_1,\partial\!\!\!/\varphi_2) \tag{1.13}$$

for any smooth section φ_1,φ_2 on $\mathcal{S}_n$.

Proof. Let $x\in M$, and choose a orthonormal local frame e_i, we then have $\nabla e_i = 0$ at x .
We then have

$$\begin{aligned}
\langle\partial\!\!\!/\varphi_1,\varphi_2\rangle &= \langle e_i\nabla_{e_i}\varphi_1,\varphi_2\rangle && (1.14)\\
&= -\langle\nabla_{e_i}\varphi_1,e_i\varphi_2\rangle && (1.15)\\
&= -e_i\langle\varphi_1,e_i\varphi_2\rangle + \langle\varphi_1,\nabla_{e_i}(e_i\varphi_2)\rangle && (1.16)\\
&= -e_i\langle\varphi_1,e_i\varphi_2\rangle + \langle\varphi_1,\nabla_{e_i}(e_i\varphi_2)\rangle && (1.17)
\end{aligned}$$

then at x, we have

$$\begin{aligned}
\langle\partial\!\!\!/\varphi_1(x),\varphi_2(x)\rangle &= -e_i\langle\varphi_1(x),e_i\varphi_2(x)\rangle + \langle\varphi_1(x),\nabla_{e_i}(e_i\varphi_2)(x)\rangle && (1.18)\\
&= -e_i\langle\varphi_1(x),e_i\varphi_2(x)\rangle + \langle\varphi_1(x),e_i\nabla_{e_i}\varphi_2(x)\rangle && (1.19)\\
&= -e_i\langle\varphi_1(x),e_i\varphi_2(x)\rangle + \langle\varphi_1(x),\partial\!\!\!/\varphi_2(x)\rangle. && (1.20)
\end{aligned}$$

We now consider $V^i = \langle \varphi_1, e_i\varphi_2\rangle$ as the i^{th} component of a vector field V(in fact V is a complexified vector field, i.e. a section of $TM \otimes \mathbb{C}$). The preceding formula then becomes

$$\langle \not{\partial}\varphi_1, \varphi_2\rangle = -divV + \langle \varphi_1, \not{\partial}\varphi_2\rangle. \tag{1.21}$$

□

Corollary 1.6. *On a compact spin manifold M, $\not{\partial}\varphi = 0$ for a smooth section φ on $\mathcal{S}_n$ if and only if $\not{\partial}^2\varphi = 0$.*

Theorem 1.7. *Let M be a spin manifold with a local orthonormal frame field $\{e_i\}$. Then the Dirac operator $\not{\partial}$ satisfies*

$$\not{\partial}^2 = -\nabla_{e_i}\nabla_{e_i} + \frac{1}{4}R, \tag{1.22}$$

where R is the scalar curvature of M.

Proof. Let $x \in M$, and choose a orthonormal local frame e_i, we then have $\nabla e_i = 0$ at x .

We compute, for a spinor field φ, at x,

$$\begin{aligned}
\not{\partial}^2\varphi &= e_j\nabla_{e_j}((e_i\nabla_{e_i})\varphi) && (1.23)\\
&= e_je_i\nabla_{e_j}\nabla_{e_i}\varphi && (1.24)\\
&= -\nabla_{e_i}\nabla_{e_i}\varphi + \sum_{i<j} e_je_i(\nabla_{e_j}\nabla_{e_i} - \nabla_{e_i}\nabla_{e_j})\varphi && (1.25)\\
&= -\nabla_{e_i}\nabla_{e_i}\varphi + \sum_{i<j} e_je_iR(e_j, e_i)\varphi && (1.26)
\end{aligned}$$

Recall that

$$R(e_i, e_j) = \frac{1}{2}\sum_{k<l}\langle R(e_i, e_j)e_k, e_l\rangle e_ke_l, \tag{1.27}$$

then

$$\frac{1}{2}\sum_{i<j}\sum_{k<l}\langle R(e_i, e_j)e_k, e_l\rangle e_ie_je_ke_l = \frac{1}{8}\sum_{i,j,k,l}\langle R(e_i, e_j)e_k, e_l\rangle e_ie_je_ke_l, \tag{1.28}$$

If i, j, k are all distinct,

$$e_ie_je_k = e_je_ke_i = e_ke_ie_j$$

, and the first Bianchi identity implies in this case that

$$R(e_i, e_j)e_k + R(e_j, e_k)e_i + R(e_k, e_i)e_j = 0$$

. The remaining terms are

$$\begin{aligned}
\frac{1}{8}\sum_{i,j,l}(\langle R(e_i, e_j)e_i, e_l\rangle e_ie_je_ie_l + R(e_i, e_j)e_j, e_l\rangle e_ie_je_je_l) &= -\frac{1}{4}\sum_{i,j,l}\langle R(e_i, e_j)e_j, e_l\rangle e_ie_l && (1.29)\\
&= \frac{1}{4}R. && (1.30)
\end{aligned}$$

$\square$

2. PROOF OF THE THE LICHNEROWICZ VANISHING THEOREM

Let (M, g) be a connected, compact, oriented and even dimensional spin manifold, the scalar curvature R of M is nonnegative and not identically zero. By lemma1.3, we can define $\not{\partial}^+$ as the restriction of $\not{\partial}$ to $\Gamma(\mathcal{S}_n^+)$, and $\not{\partial}^-$ similarly. Then *index*$(\not{\partial}^+) = \dim\ker\not{\partial}^+ - \dim\ker\not{\partial}^-$.

Theorem 2.1. *Let (M, g) be a connected, compact, oriented and even dimensional spin manifold, the scalar curvature R of M is nonnegative and not identically zero, then $\hat{A}(M) = 0$, where $\hat{A}(M)$ is a topological quantity.*

Proof.

$$\begin{aligned} 0 &= \int_M \not{\partial}^2\varphi\cdot\varphi && (2.1)\\ &= \int_M -\nabla_{e_i}\nabla_{e_i}\varphi\cdot\varphi + \frac{1}{4}R\varphi^2 && (2.2)\\ &= \int_M |\nabla\varphi|^2 + \frac{1}{4}R\varphi^2 && (2.3)\end{aligned}$$

$\Rightarrow \nabla\varphi = 0$, i.e. φ is parallel. Since the scalar curvature R of M is nonnegative and not identically zero, then $\varphi = 0$. $\square$